Constructing three and higher n-dimensional complex number spherical and Cartesian coordinate systems based on rotation factors

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Abstract

Based on the set of real numbers and the set of rotation factors, the constructions of three and higher n-dimensional complex number spherical coordinate systems are realized. The projections of complex numbers or position vectors from four-dimensional space to three-dimensional space as well as from a higher n-dimensional space to a lower dimensional space are conceived. The projections are consistent throughout regardless of the dimensional levels and the generalization of the n-dimensional coordinate systems are achieved with n=2 for two-dimensional plane, n=3 for three-dimensional space, n=4 for four-dimensions and so on. The generalization of transformation to n-dimensional Cartesian coordinate systems from the spherical systems are obtained. The rotation operations in the n-dimensional complex number spherical coordinate systems are succinct and efficient, and the results can be transformed back to Cartesian coordinate systems, and vice versa.

1. Introduction

Hypercomplex numbers [3] are an extension to higher dimensions from the standard two-dimensional complex numbers. One established example of hypercomplex numbers is the quaternion number system by Hamilton [2], which is in four dimensions and non-commutative for multiplication. There exist other hypercomplex systems [6–8]. These hypercomplex systems tend to be in Cartesian coordinate systems that are inconvenient for handling rotation operations.

Furthermore, in general, there are restrictions with respect to commutative and associative properties of hypernumbers in three and other higher dimensions, Ferdinand Georg Frobenius [1, 4] proved that for a division algebra [4] over the real numbers to be finite-dimensional and associative, it cannot be three-dimensional, and there are only three such division algebras: real numbers, complex numbers, and quaternions, which have dimension 1, 2, and 4 respectively.

The recent discovery of the causality origin of imaginary unit and the existence of rotation factors by the author [5] provides a novel approach to construct complex number coordinate systems for hypercomplex numbers, and gives rise to complex number spherical coordinate systems that are generalizable to n-dimensions from 1, 2, 3, 4 to a higher positive-integer n. This paper presents the construction of three-dimensional complex number spherical coordinate system and extends the methodology to higher n-dimensional spaces. The results of efficient rotation operations conducted in the spherical coordinate systems can be transformed back to Cartesian coordinate systems, and vice versa.

2. Position numbers and rotation factors in relationship with complex numbers

2.1. Position vectors, position numbers, and rotation factors

In a two-dimensional plane, a point may be represented by a position vector, which has a direction from the coordinate origin to the point itself, and has a magnitude that is equal to the distance between the point to the origin. By the same token, this position vector concept can be applied to three and higher n-dimensional spaces.

In the paper [5], for a two-dimensional plane, position numbers are introduced to represent points in the plane. A position number representing a point has a direction that is also from the coordinate origin to the point itself. The direction of a position number may be implicit or implied without being explicitly denoted by an arrow, as there is no ambiguity with respect to the implied direction. A position number, a position vector, or a vector formed by two points in the plane, can be rotated by an angle specific rotation factor. Specifically, the formula for a rotation factor q of angle θ is

$$\mathrm{q}(\theta) = e^{i\theta} \tag{1}$$

where i is the orthogonal rotation factor (rotation factor of angle $\frac{\pi}{2}$) and is equivalent to the imaginary unit in the 2D complex plane. Multiplying a rotation factor $\mathrm{q}(\theta)$ or $e^{i\theta}$ to a position number or a position vector makes the position point rotate counter-clockwise by angle θ with the magnitude of the position number or the position vector unchanged. A rotation factor has a unit magnitude of 1. It is noted that for a two-dimensional plane, position numbers are equivalent to complex numbers.

All real numbers form a field [10]. The set of real numbers is denoted $\mathbb{R}$ [9]. The set of rotation factors is defined as

$$\mathbb{E} = \{e^{i\theta} \mid \theta \in \mathbb{R}, 0 \leq \theta \leq 2\pi\} \tag{2}$$

The concepts of position vectors, position numbers, and rotation factors may be extended to three and higher n-dimensional spaces.

2.2. Terminologies and conventions

Space: refers to Euclidean space.

Position vector: A position vector represents the position of a point in an n-dimensional space with n=1 for the position in a one-dimensional axis, n=2 for the position in a two-dimensional plane, and n=3 or higher for the position in a three or higher n-dimensional space. The direction of a position vector is from the coordinate origin to the point represented by the vector, and the magnitude of the vector is the distance between the point and the origin.

Position number: Almost identical to a position vector in the representation sense, a position number represents the position of a point in an n-dimensional space with n=1 for the position in a one-dimensional axis, n=2 for the position in a two-dimensional plane, and n=3 or higher for the position in a three or higher n-dimensional space. The direction of a position number is from the coordinate origin to the point represented by the number, and the magnitude of the number is the distance between the point and the origin.

Position vector and position number conventions: Here the terms position vector and position number are interchangable for representing a point. For geometric representation, vector arrow may be used for a position number to explicitly indicate the direction of the position number.

Point of position vector or position number: means the point represented by the position vector or the position number, and vice versa.

Direction of point, position vector, or position number: means the same direction as the direction from the origin point to the point.

Rotating (or rotation of) point, position vector, or position number: means the same as rotating a point to another position with respective position vector change or position number change.

Applying a rotation factor to: means multiplying a rotation factor to a position vector or a position number. Applying a rotation factor to a point means applying the factor to the point's position vector or position number.

Target of a rotation factor: means the position vector or the position number that a rotation factor is applied to.

Position numbers, complex numbers, hypercomplex numbers: A position number may also be called a complex number in a two-dimensional plane, or called a hypercomplex number in a higher dimensional space. In essence, complex numbers and hypercomplex numbers represent the positions of points in their respective planes or spaces. The position concept is commonly known and clear. In this sense, the term, position number is mainly used here.

3. Position numbers in three-dimensional space

In [5], based on the set of real numbers $\mathbb{R}$ and the set of rotation factors $\mathbb{E}$ (2), the construction of two-dimensional polar and Cartesian coordinate systems has been realized. Next, with the same approach, the construction of three-dimensional spherical and Cartesian coordinate systems by rotation factors is presented.

3.1. Position numbers in spherical coordinate system

In Figure 1, RE denotes a real number axis. Point O is the origin. OR_i is an axis that is orthogonal to the RE axis. Applying the orthogonal rotation factor $i = e^{i\frac{\pi}{2}}$ to any point in the RE axis makes the point rotate counter-clockwise by angle $\frac{\pi}{2}$ and reach the OR_i axis. Thus, the i indicates the direction of the OR_i axis, and OR_i may also be called i axis. A rotation factor $e^{i\theta}$ of an arbitrary angle θ may be called i-based rotation factor as i is present in the factor. The i and i-based rotation factor $e^{i\theta}$ are associated with the OR_i axis.

The OR_j axis is orthogonal to the 2nd dimension axis-plane, which is formed by the RE axis and the OR_i axis. OR_j has the direction of the cross product of the RE direction unit vector and the OR_i direction unit vector (right-hand rule). Applying orthogonal rotation factor j to any point in the 2nd dimension axis-plane makes the point rotated to the OR_j axis. Thus, the j indicates the direction of the OR_j axis and is associated with the axis. OR_j may also be called j axis.

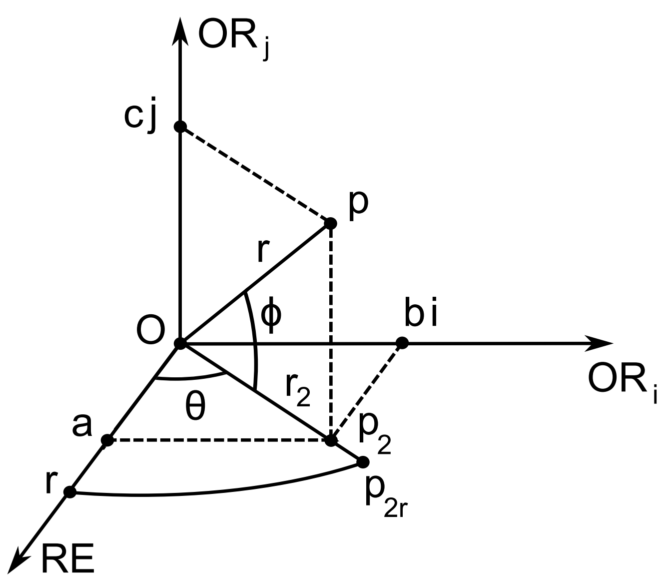

Figure 1. Construction of three-dimensional spherical and Cartesian coordinate systems by rotation factors

Number r is a real number in the RE axis and represents the corresponding point in the axis. Applying an i-based rotation factor $e^{i\theta}$ to number r makes the point rotate by angle

θ to position number p_{2r} in the plane formed by the RE axis and the OR_i axis. Thus, the position number p_{2r} is expressed as

$$p_{2r} = re^{i\theta} \tag{3}$$

Applying a j-based rotation factor $e^{j\phi}$ to position number p_{2r} makes the point rotate by angle ϕ to position number p in the plane formed by the p_{2r} point and the OR_j axis. That is, $p = p_{2r}e^{j\phi}$. With this and (3), the position number p becomes

$$p = re^{i\theta}e^{j\phi} \tag{4}$$

Equation (4) gives the formula for three-dimensional position numbers in the spherical coordinate system. The formula is surprisingly succinct and elegant.

3.2. Position numbers in Cartesian coordinate system

Next, the formula for three-dimensional position numbers in the Cartesian coordinate system is derived.

In Figure 1, the coordinate for a specific point in the OR_j axis is denoted by cj, where j is the orthogonal rotation factor that indicates the direction of the OR_j axis and c is a real number. The line formed by the point cj and the point p is parallel to the line formed by the origin O and the point p_{2r}.

The point represented by position number p_2 is in the plane formed by the RE axis and the OR_i axis. The line formed by the point p_2 and the point p is parallel to the OR_j axis. The position number p_2 has a magnitude of r_2.

The coordinate for a specific point in the RE axis is denoted by a real number a. The line formed by the point a and the point p_2 is parallel to the OR_i axis.

Similarly, the coordinate for a specific point in the OR_i axis is denoted by bi, where i is the orthogonal rotation factor that indicates the direction of the OR_i axis and b is a real number. The line formed by the point bi and the point p_2 is parallel to the RE axis.

The above indicates that the position number p has a coordinate of a in the RE axis, a coordinate of bi in the OR_i axis, and a coordinate of cj in the OR_j axis. That is

$$p = a + bi + cj \tag{5}$$

Equation (5) represents three-dimensional position numbers in Cartesian coordinate system.

The geometry in Figure 1 gives that $c = rsin(\phi)$, $r_2 = rcos(\phi)$, $a = r_2cos(\theta)$, and $b = r_2sin(\theta)$, which lead to

$$a = rcos(\theta)cos(\phi) \tag{6}$$

$$\text{b} = rsin(\theta)cos(\phi) \tag{7}$$

$$\text{c} = rsin(\phi) \tag{8}$$

Equations (6), (7), and (8) represent the transformation from spherical coordinates to Cartesian coordinates.

With the transformation, Equation (5) becomes

$$\text{p} = r(cos(\theta)cos(\phi) + isin(\theta)cos(\phi) + jsin(\phi)) \tag{9}$$

From Equations (6), (7), and (8), the transformation from Cartesian coordinates to spherical coordinates is represented by

$$r = \sqrt{a^2 + b^2 + c^2} \tag{10}$$

$$\theta = tan^{-1}(\frac{\text{b}}{\text{a}}) \tag{11}$$

$$\phi = sin^{-1}(\frac{\text{c}}{\text{r}}) \tag{12}$$

3.3. Rotation operations being commutative

In Figure 1, the rotation factor $e^{i\theta}$ is first applied to the number r in the RE axis, and then is followed by the rotation factor $e^{j\phi}$.

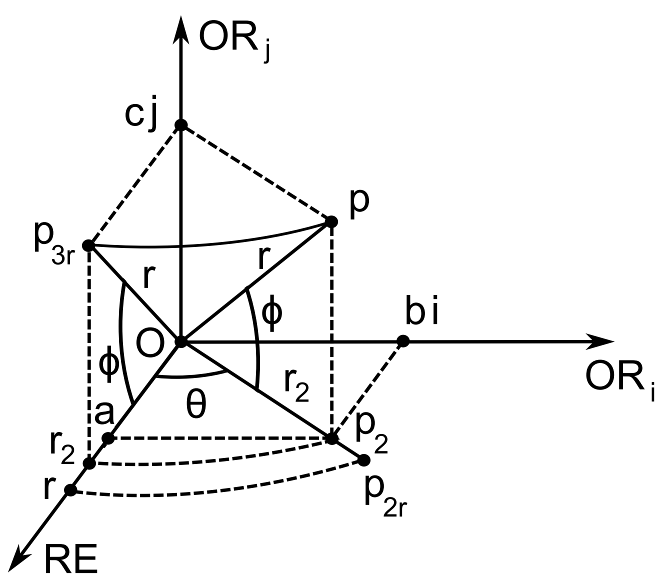

Figure 2. The order of rotation operations being reversed

Figure 2 is the same as Figure 1, except that the order of the operations by the two rotation factors is reversed.

In Figure 2, applying the rotation factor $e^{j\phi}$ to number r in the RE axis makes the point rotate by angle ϕ to position number $\mathrm{p_{3r}}$ in the plane formed by the RE axis and the $\mathrm{OR_j}$ axis. The point represented by number $\mathrm{r_2}$ and the point represented by $\mathrm{p_{3r}}$ form a line that is parallel to the $\mathrm{OR_j}$ axis. Then, applying the rotation factor $e^{i\theta}$ to position number $\mathrm{p_{3r}}$ makes the point rotate by angle θ to position number p in the plane that is parallel to the plane formed by the RE axis and the $\mathrm{OR_i}$ axis. The final position is the point p, which is the same as the point in Figure 1. That is

$$\mathrm{p} = re^{i\theta}e^{j\phi} = re^{j\phi}e^{i\theta} \tag{13}$$

Equation (13) represents the two rotation factor operations being commutative.

3.4. Interaction between orthogonal rotation factors i and j

In Figure 1, the construction of the spherical and Cartesian has not involved the interaction between the orthogonal rotation factors i and j. The result in (9) contains the interaction information. By Euler's formula, $i = e^{i\frac{\pi}{2}}$ and $j = e^{j\frac{\pi}{2}}$. Then $ij = e^{i\frac{\pi}{2}}e^{j\frac{\pi}{2}}$ becomes a position number with r=1, $\theta = \frac{\pi}{2}$, $\phi = \frac{\pi}{2}$. Inserting the values into Equation (9) gives $\mathrm{p} = j$. That is

$$ij = j \tag{14}$$

By the same token, $ji = e^{j\frac{\pi}{2}}e^{i\frac{\pi}{2}}$ is a position number with r=1, $\theta = \frac{\pi}{2}$, $\phi = \frac{\pi}{2}$. Equation (9) gives $\mathrm{p} = j$. That is

$$ji = j = ij \tag{15}$$

The results in (14) and (15) can also be directly obtained from the geometric representation. In Figure 2, applying the orthogonal rotation factor j to any point in the i axis makes the point rotate to the j axis. The point represented by i is in the i axis. Thus, $ij = j$. In the same figure, on the other hand, applying the orthogonal rotation factor i to the point $\mathrm{p_{3r}}$ makes the point rotate to the plane formed by the $\mathrm{OR_i}$ axis and the $\mathrm{OR_j}$ axis. But if $\phi = \frac{\pi}{2}$, $\mathrm{p_{3r}}$ is in the j axis and applying i to $\mathrm{p_{3r}}$ will not move the point. Thus, $\mathrm{p_{3r}}i = \mathrm{p_{3r}}$. Let $\mathrm{p_{3r}} = j$, it follows that that $ji = j$.

In fact, any point in the plane formed by the RE axis and the $\mathrm{OR_i}$ axis will rotate to the $\mathrm{OR_j}$ axis if the point is applied by the rotation factor $j = e^{j\frac{\pi}{2}}$. In Figure 1, if $\phi = \frac{\pi}{2}$, point $\mathrm{p_{2r}}$ reaches the $\mathrm{OR_i}$ axis. That is, $\mathrm{p_{2r}}e^{j\frac{\pi}{2}} = \mathrm{p_{2r}}j = |\mathrm{p_{2r}}|j$. Particularly, $e^{j\theta}$ is in the plane and has a unit magnitude. It will also rotate to the $\mathrm{OR_i}$ axis if it is applied by j. That is

$$e^{i\theta}j = j \tag{16}$$

The result in (16) can also be obtained through Equation (9) by the coordinate transformation for position number $e^{i\theta}j = e^{i\theta}e^{j\frac{\pi}{2}}$ with r=1, $\phi = \frac{\pi}{2}$.

3.5. Obtaining spherical-to-Cartesian transformation by rotation factor multiplication algebra

A position number is represented by Equation (4) in spherical coordinate system. The transformation of the position number into Cartesian coordinates is presented next. With Euler's formula for the rotation factor $e^{j\phi}$ in (4), it follows that

$$\mathrm{p} = re^{i\theta}(cos(\phi) + jsin(\phi)) \tag{17}$$

With multiplication algebra, Equation (17) becomes

$$\mathrm{p} = r(e^{i\theta}cos(\phi) + e^{i\theta}jsin(\phi)) \tag{18}$$

And with $e^{i\theta}j = j$ in (16) and Euler's formula for $e^{i\theta}$, Equation (18) leads to

$$\mathrm{p} = r(cos(\theta)cos(\phi) + isin(\theta)cos(\phi) + jsin(\phi)) \tag{19}$$

Equation (19) is the same as Equation (4). That is, the spherical-to-Cartesian transformation has been achieved through rotation factor multiplication algebra.

3.6. Multiplication of position numbers

In spherical coordinate system represented by (4), let $\mathrm{p}_1 = \mathrm{r}_1 e^{i\theta_1}e^{j\phi_1}$ be one position number, and $\mathrm{p}_2 = \mathrm{r}_2 e^{i\theta_2}e^{j\phi_2}$ be another.

As previously shown, the two rotation factors, $e^{i\theta}$ and $e^{j\phi}$ are commutative for rotating a position number or a position vector for that matter. The rotation of position number p_1 by p_2 through its rotation factors, $e^{i\theta_2}$ and $e^{j\phi_2}$ is commutative, and vice versa for the rotation of position number p_2 by p_1. And r_1 and r_2 are positive real numbers and are commutative in the multiplication of p_1 and p_2. That is, $\mathrm{p}_1 \cdot \mathrm{p}_2 = \mathrm{p}_2 \cdot \mathrm{p}_1$.

The multiplication of the two position numbers is

$$\mathrm{p}_1 \cdot \mathrm{p}_2 = \mathrm{r}_1 e^{i\theta_1}e^{j\phi_1}\mathrm{r}_2 e^{i\theta_2}e^{j\phi_2} \tag{20}$$

With the rotation factors and the positive real numbers being commutative, it follows that

$$\mathrm{p}_1 \cdot \mathrm{p}_2 = \mathrm{p}_2 \cdot \mathrm{p}_1 = \mathrm{r}_1\mathrm{r}_2 e^{i(\theta_1+\theta_2)}e^{j(\phi_1+\phi_2)} \tag{21}$$

Equation (21) indicates that the multiplication result is a position number with a magnitude of $\mathrm{r}_1\mathrm{r}_2$, rotation factors $e^{i\theta'}$ and $e^{j\phi'}$ where $\theta' = \theta_1 + \theta_2$ and $\phi' = \phi_1 + \phi_2$, as the sum of the respective angles of rotation factors.

With (9), Equation (21) can be transformed to Cartesian coordinates as

$$\mathrm{p}_1 \cdot \mathrm{p}_2 = \mathrm{r}_1\mathrm{r}_2(cos(\theta_1+\theta_2)cos(\phi_1+\phi_2) + isin(\theta_1+\theta_2)cos(\phi_1+\phi_2) + jsin(\phi_1+\phi_2)) \tag{22}$$

In Cartesian coordinate system represented by (5), assume that one first has the position numbers, p_1 and p_2 in Cartesian coordinates as $p_1 = a_1 + b_1 i + c_1 j$, and $p_2 = a_2 + b_2 i + c_2 j$. For the multiplication of $p_1 \cdot p_2$, p_1 and p_2 may be first transformed into spherical coordinates as $p_1 = r_1 e^{i\theta_1} e^{j\phi_1}$, and $p_2 = r_2 e^{i\theta_2} e^{j\phi_2}$. The multiplication is then performed in the spherical system, and the result for Cartesian coordinates is given by Equation (22).

3.7. Vector rotations by rotation factors with coordinate system independence

It is important to note that rotation factors can be independent of coordinate systems and can encourage one's creativity for the novel utilization to achieve desired results.

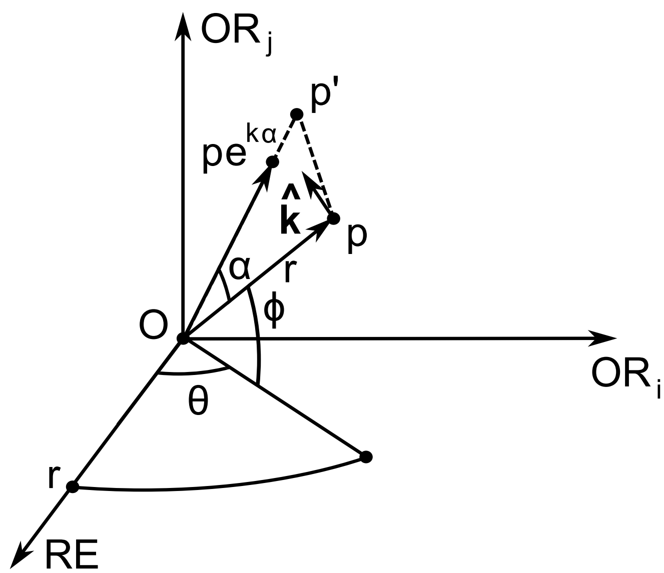

Figure 3. Vector rotation by rotation factor unassociated with the coordinate axes

With the same notations as those in Figure 1, Figure 3 illustrates a vector rotation by rotation factor that is not associated with the coordinate axes.

A position number is equivalent to a position vector with the same direction and magnitude. In Figure 3, position numbers have been denoted with direction arrows and may also viewed as the position vectors. The position number p now has been represented by a vector with the direction pointing from the origin O to the point. Assume that there is another point p′, and the angle between the position vectors of the two points p and p′ is α. One wants to rotate the point p to the direction of the point p′.

Next a rotation factor is constructed to perform the rotation. The origin O, the point p and the point p′ form a plane. $\hat{\mathbf{k}}$ is a unit vector in the plane and is orthogonal to the target vector p. The rotation factor $e^{k\alpha}$ has a k orthogonal rotation factor that is corresponding to the direction of the unit vector $\hat{\mathbf{k}}$. Applying the rotation factor $e^{k\alpha}$ to position number p makes it rotate to the point represented by position number $pe^{k\alpha}$, which has the same direction of the point p′.

4. Projections of position vectors in n-dimensional spherical and Cartesian coordinate systems

Before extending results obtained from a three dimensional space to a higher n-dimensional space, one may need to understand the projections from an n-dimensional space to a lower dimensional space.

If the three dimensional coordinate space in Figure 1 is projected to a two-dimensional plane from the direction of the RE axis towards the plane, Figure 4 illustrates the projection result.

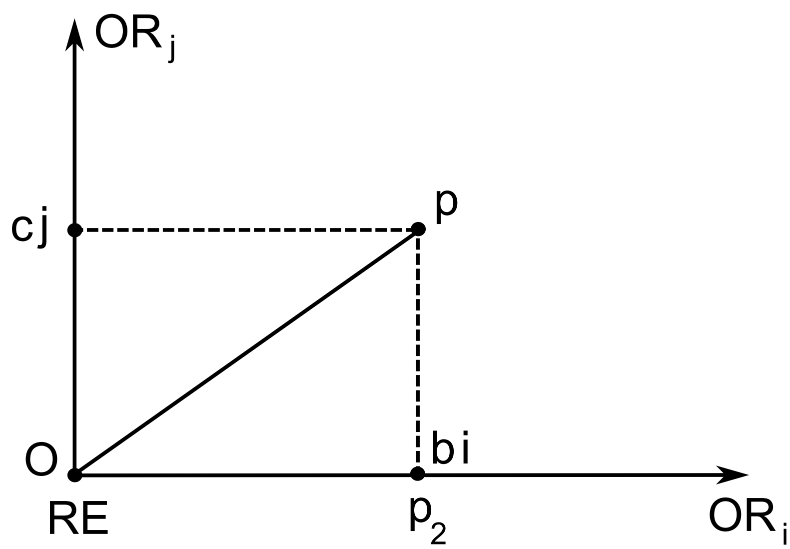

Figure 4. Projection of a three-dimensional coordinate space onto a two-dimensional plane

After the projection, the respective level of a three-dimensional (3D) space, a two-dimensional (2D) plane, or a one-dimensional (1D) line is lowered by 1. The 3D coordinate space appears 2D on the projection plane, and the 2D plane formed by the RE axis and the $\mathrm{OR_i}$ axis appears 1D. The 1D line of the RE axis appears as one point and overlaps with the origin O.

Next, the projections of position vectors from a higher dimension to a lower dimension are presented and discussed.

4.1. 2D plane

Figure 5 illustrates the projections of a position vector in a 2D plane. In the figure, O is the origin. RE is a real number axis. $\mathrm{OR_2}$ is an axis that is orthogonal to the RE axis. Applying orthogonal rotation factor $i_2 = e^{i_2 \frac{\pi}{2}}$ to any point in the RE axis makes the point rotate counter-clockwise by angle $\frac{\pi}{2}$ and reach the $\mathrm{OR_2}$ axis. Thus, the i_2 indicates the direction of the OR_2 axis. For a regular 2D complex number plane, i_2 is considered as the imaginary unit i.

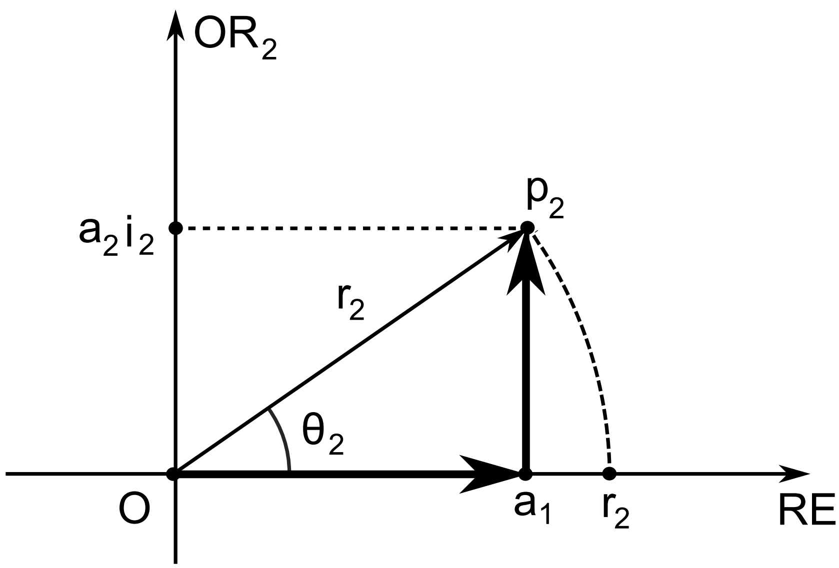

Figure 5. Projections of a position vector in a 2D plane

Applying an i_2-based rotation factor $e^{i_2\theta_2}$ of angle θ_2 to the point represented by real number r_2 in the RE axis makes the point rotate to the point represented by position number p_2. The r_2 is the magnitude of p_2.

The vectors with bold arrows in the figure denote the projections. The projection of the position vector represented by p_2 to the lower 1D dimension is the position number a_1 or the vector represented by a_1 with $a_1 = r_2 cos(\theta_2)$. The projection of the position vector to the direction of the orthogonal axis OR_2 is $a_2 i_2$ or the vector from the point a_1 to the point p_2 with $a_2 = r_2 sin(\theta_2)$. It is noted that a_1 is considered as a position number p_1 in a lower 1D coordinate system.

The rotation and projections in the figure lead to

$$p_2 = r_2 e^{i_2\theta_2} \tag{23}$$

$$p_2 = p_1 + a_2 i_2 \tag{24}$$

$$p_1 = a_1 = r_2 cos(\theta_2) \tag{25}$$

$$a_2 = r_2 sin(\theta_2) \tag{26}$$

The 2D equation set of (23)-(26) represent spherical and Cartesian coordinate systems in a 2D plane.

4.2. 3D space

Compared with the 2D plane in Figure 5, one more orthogonal axis OR_3 is added for the coordinate system in a 3D space as illustrated in Figure 6. The OR_3 axis is orthogonal to the 2nd dimension axis-plane formed by the RE and OR_2 axes in the lower dimensions.

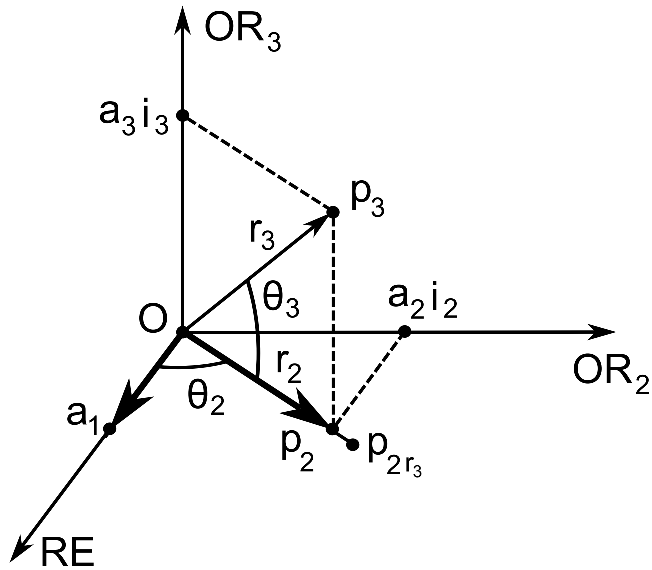

Figure 6. Projections of a position vector in a 3D space

The point represented by position number $\mathrm{p}_{2\mathrm{r}_3}$ is in the same plane as p_2 and also in the same direction. $\mathrm{p}_{2\mathrm{r}_3}$ has a magnitude of r_3. From (23), $\mathrm{p}_2 = \mathrm{r}_2 e^{i_2\theta_2}$. Thus, $\mathrm{p}_{2\mathrm{r}_3} = \mathrm{r}_3 e^{i_2\theta_2}$. Applying an i_3-based rotation factor $e^{i_3\theta_3}$ of angle θ_3 to the point represented by $\mathrm{p}_{2\mathrm{r}_3}$ makes the point rotate to the point represented by position number p_3. Thus, $\mathrm{p}_3 = \mathrm{r}_3 e^{i_2\theta_2} e^{i_3\theta_3}$. The r_3 is the magnitude of p_3.

The vectors with bold arrows in the figure denote the projections. The position vector represented by p_2 in Figure 5 now becomes a projection in the current 3D space. The projection of the p_3 vector to the lower 2D dimension is the position vector of p_2 with $\mathrm{r}_2 = \mathrm{r}_3 cos(\theta_3)$. The projection of the p_3 vector to the direction of the orthogonal axis OR_3 is $\mathrm{a}_3\mathrm{i}_3$ with $\mathrm{a}_3 = \mathrm{r}_3 sin(\theta_3)$.

The above rotation and projections lead to

$$\mathrm{p}_3 = \mathrm{r}_3 e^{i_2\theta_2} e^{i_3\theta_3} \tag{27}$$

$$\mathrm{p}_3 = \mathrm{p}_2 + \mathrm{a}_3\mathrm{i}_3 \tag{28}$$

$$\mathrm{r}_2 = \mathrm{r}_3 cos(\theta_3) \tag{29}$$

$$a_3 = r_3 sin(\theta_3) \tag{30}$$

The 3D equation set of (27)-(30) represent spherical and Cartesian coordinate systems in a 3D space, with p_2 for the lower dimension 2D given by the 2D equation set (23)-(26).

4.3. 4D space

Similarly, compared with the 3D plane in Figure 6, one more orthogonal axis OR_4 is added for the coordinate system in a 4D space as illustrated in Figure 7.

It is noted that Figure 7 is viewed from the 4D perspective. Logically, if the 4D space occupies our 3D space and the 3D space is one dimension lower than the 4D, the 3D space must appear as a 2D plane from the 4D viewpoint.

In the 4D space, the OR_4 axis is orthogonal to the 3rd dimension axis-plane, which is formed by the OR_2 and OR_3 axes in the lower dimensions. OR_4 has the direction that follows the right-hand rule with respect to the OR_2 and OR_3 axes.

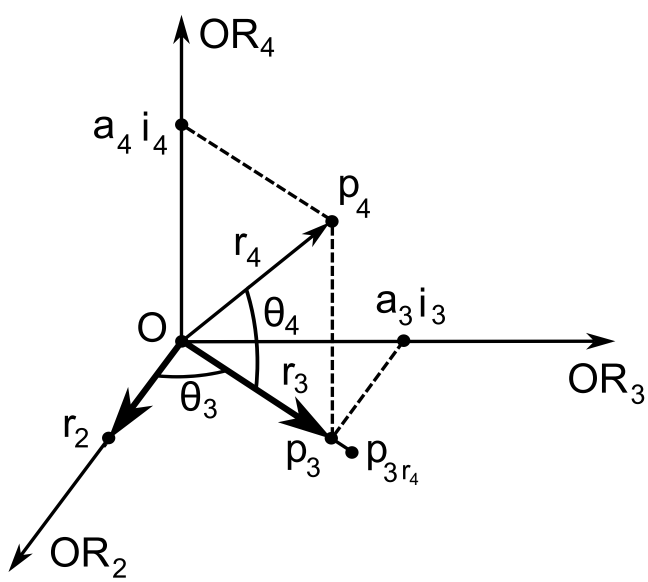

Figure 7. Projections of a position vector in a 4D space

The point represented by position number p_{3r_4} is in the same plane as p_3 and also in the same direction. p_{3r_4} has a magnitude of r_4. From (27) for p_3, it follows that $p_{3r_4} = r_4 e^{i_2\theta_2} e^{i_3\theta_3}$. Applying an i_4-based rotation factor $e^{i_4\theta_4}$ of angle θ_4 to the point represented by p_{3r_4} makes the point rotate to the point represented by position number p_4. Thus, $p_4 = r_4 e^{i_2\theta_2} e^{i_3\theta_3} e^{i_4\theta_4}$. The r_4 is the magnitude of p_4.

The vectors with bold arrows in the figure denote the projections. The position vector represented by p_3 in Figure 6 now becomes a projection in the current 4D space. The projection of the p_4 vector to the lower 3D dimension is the position vector of p_3 with $r_3 = r_4 cos(\theta_4)$. The projection of the p_4 vector to the direction of the orthogonal axis OR_4 is $a_4 i_4$ with $a_4 = r_4 sin(\theta_4)$.

The above rotation and projections lead to

$$p_4 = r_4 e^{i_2\theta_2} e^{i_3\theta_3} e^{i_4\theta_4} \tag{31}$$

$$p_4 = p_3 + a_4 i_4 \tag{32}$$

$$r_3 = r_4 cos(\theta_4) \tag{33}$$

$$a_4 = r_4 sin(\theta_4) \tag{34}$$

The 4D equation set of (31)-(34) represent spherical and Cartesian coordinate systems in a 4D space, with p_3 for the lower dimension 3D given by the 3D equation set (27)-(30).

4.4. n-dimensional space

By the same token, Figure 8 shows the projections in an n-dimensional space.

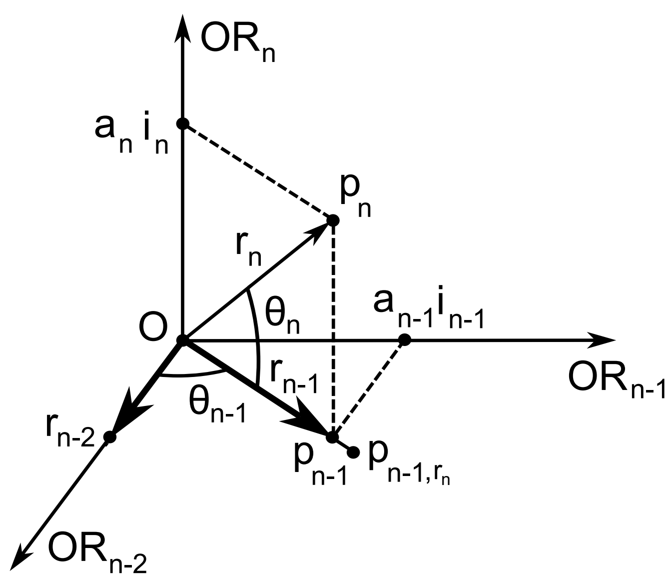

Figure 8. Projections of a position vector in an n-dimensional space

From the n-dimensional space perspective, the n-1 dimensional space appears as a plane in the figure. The OR_n axis is orthogonal to the (n-1)th dimension axis-plane, which is formed by the OR_{n-2} and OR_{n-1} axes in the lower dimensions. OR_n has the direction that follows the right-hand rule with respect to the OR_{n-2} and OR_{n-1} axes.

The point represented by position number p_{n-1,r_n} is in the same plane as p_{n-1} and also in the same direction. p_{n-1,r_n} has a magnitude of r_n. Applying an i_n-based rotation factor $e^{i_n\theta_n}$ of angle θ_n to the point represented by p_{n-1,r_n} makes the point rotate to the point represented by position number p_n.

The vectors with bold arrows in the figure denote the projections. The projection of the p_n vector to the lower (n-1)th dimension is the position vector of p_{n-1} with $r_{n-1} = r_n cos(\theta_n)$. The projection of the p_n vector to the direction of the orthogonal axis OR_n is $a_n i_n$ with $a_n = r_n sin(\theta_n)$.

The above rotation and projections lead to

$$p_n = r_n e^{i_2\theta_2} e^{i_3\theta_3} e^{i_4\theta_4} ... e^{i_n\theta_n} \tag{35}$$

$$p_n = p_{n-1} + a_n i_n \tag{36}$$

$$r_{n-1} = r_n cos(\theta_n) \tag{37}$$

$$a_n = r_n sin(\theta_n) \tag{38}$$

The n-dimensional equation set of (35)-(38) represent spherical and Cartesian coordinate systems in an n-dimensional space, with p_{n-1} for the lower dimension given by the corresponding n-1 equation set.

5. Generalization of n-dimensional spherical and Cartesian coordinate systems

5.1. n-dimensional spherical coordinate system

For the generalization of the coordinate system construction, in the 2D plane, the orthogonal axis OR_2 is orthogonal to the RE real number axis. The associated rotation factor for the OR_2 axis is $e^{i_2\theta_2}$.

Then, in the 3D space, the orthogonal axis OR_3 is introduced to be orthogonal to the 2nd-dimension axis-plane formed by the RE and OR_2 axes. The associated rotation factor for the OR_3 axis is $e^{i_3\theta_3}$.

Further, in the 4D space, the orthogonal axis OR_4 is introduced to be orthogonal to the 3rd-dimension axis-plane formed by the OR_2 and OR_3 axes. The associated rotation factor for the OR_4 axis is $e^{i_4\theta_4}$.

Finally, in the n-dimensional space, the orthogonal axis $\mathrm{OR_n}$ is introduced to be orthogonal to the (n-1)th dimension axis-plane formed by $\mathrm{OR_{n-2}}$ and $\mathrm{OR_{n-1}}$ axes. The associated rotation factor for the $\mathrm{OR_n}$ axis is $e^{i_n\theta_n}$.

With (35), the position number in an n-dimensional spherical coordinate system is given by

$$\mathrm{p_n} = \mathrm{r_n}\prod_{j=2}^{n} e^{i_j\theta_j} \tag{39}$$

where $\mathrm{r_n}$ is the magnitude of the position number and $e^{i_j\theta_j} \in \mathbb{E}$ in (2).

5.2. n-dimensional Cartesian coordinate system transformed from the spherical coordinate system

From (24)-(26) for 2D, (28)-(30) for 3D and (32)-(34) for 4D, it follows that

$$\mathrm{p_2} = \mathrm{r_2}(cos(\theta_2) + i_2 sin(\theta_2)) \tag{40}$$

$$\mathrm{p_3} = \mathrm{r_3}(cos(\theta_2)cos(\theta_3) + i_2 sin(\theta_2)cos(\theta_3) + i_3 sin(\theta_3)) \tag{41}$$

$$\mathrm{p_4} = \mathrm{r_4}(cos(\theta_2)cos(\theta_3)cos(\theta_4) + i_2 sin(\theta_2)cos(\theta_3)cos(\theta_4) + i_3 sin(\theta_3)cos(\theta_4) + i_4 sin(\theta_4)) \tag{42}$$

Along with (36)-(38), the generalization of n-dimensional position number transformation from spherical coordinates to Cartesian coordinates is given by

$$\mathrm{p_n} = \sum_{j=1}^{n} a_j i_j \tag{43}$$

where $a_j = r_n E_j$ and

$$E_j = sin(\theta_j)\prod_{k=j+1}^{n} cos(\theta_k)$$

with $i_1 = 1$, $sin(\theta_1) = 1$, and $\prod_{k=j+1}^{n} cos(\theta_k) = 1$ if $j + 1 > n$.

5.3. n-dimensional spherical coordinate system transformed from the Cartesian coordinate system

For 2D, from (24)-(26), the magnitude of the position number is given by

$$\mathrm{r_2} = \sqrt{a_1^2 + a_2^2} \tag{44}$$

The tuple of each dimension specific rotation angle is given by

$$(\theta_2) = (sin^{-1}(\frac{a_2}{\mathrm{r_2}})) \tag{45}$$

Similarly, for 3D, from (28)-(30), the magnitude of the position number is given by

$$\mathrm{r}_3 = \sqrt{a_1^2 + a_2^2 + a_3^2} \tag{46}$$

The tuple of each dimension specific rotation angle is given by

$$(\theta_2, \theta_3) = (sin^{-1}(\frac{a_2}{\mathrm{r}_2}), sin^{-1}(\frac{a_3}{\mathrm{r}_3})) \tag{47}$$

And for 4D, from (32)-(34) the magnitude of the position number is given by

$$\mathrm{r}_4 = \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2} \tag{48}$$

The tuple of each dimension specific rotation angle is given by

$$(\theta_2, \theta_3, \theta_4) = (sin^{-1}(\frac{a_2}{\mathrm{r}_2}), sin^{-1}(\frac{a_3}{\mathrm{r}_3}), sin^{-1}(\frac{a_4}{\mathrm{r}_4})) \tag{49}$$

Along with (36)-(38), the generalization of n-dimensional position number transformation from Cartesian coordinates to spherical coordinates is given by

$$\mathrm{r}_\mathrm{n} = \sqrt{\sum_{j=1}^{n} a_j^2} \tag{50}$$

for the magnitude and

$$(\theta_2, \theta_3, \theta_4, ..., \theta_\mathrm{n}) \tag{51}$$

for (n-1)-tuple of angles with

$$\theta_\mathrm{j} = sin^{-1}(\frac{a_j}{\mathrm{r}_\mathrm{j}})$$

where θ_j is the rotation angle for the jth dimension with the corresponding magnitude

$$\mathrm{r}_\mathrm{j} = \sqrt{\sum_{k=1}^{j} a_k^2} \tag{52}$$

6. Generalization of multiplication algebra of n-dimensional rotation factors

The generalization of multiplication algebra of rotation factors for the coordinate system represented by Equation (39) is discussed next. The multiplication algebra involves the interactions of rotation factors and orthogonal rotation factors across different dimension levels.

6.1. 3D rotation factors

For the 3D space, in Figure 6, the point represented by the i_2-based rotation factor $e^{i_2\theta_2}$ is in the plane formed by the RE and OR_2 axes. Applying the i_3 orthogonal rotation factor to the point makes it rotate to the OR_3 axis. Thus, $e^{i_2\theta_2}i_3 = i_3$. On the other hand, as discussed in section 3.4, applying the rotation factor $e^{i_2\theta_2}$ to a point represented by i_3 in the OR_3 axis will not move the point. Thus, $i_3e^{i_2\theta_2} = i_3$. From the above, it follows that

$$e^{i_2\theta_2}i_3 = i_3e^{i_2\theta_2} = i_3 \tag{53}$$

For $\theta_2 = \frac{\pi}{2}$, Equation (53) leads to

$$i_2i_3 = i_3i_2 = i_3 \tag{54}$$

Equations (53) and (54) indicate that the orthogonal rotation factor i_3 is multiplication commutative with the orthogonal rotation factor i_2 and the rotation factor $e^{i_2\theta_2}$ in the lower dimension, and the multiplication result is always equal to the i_3 itself.

6.2. 4D rotation factors

In Figure 7 for the 4D space, the 3D space in Figure 6 now appears 2D as the projection from the 4D perspective. The OR_4 axis is orthogonal to the 3rd-dimension axis-plane, which is formed by the OR_2 and OR_3 axes. The point represented by the i_3-based rotation factor $e^{i_3\theta_3}$ is in the plane. Applying the i_4 orthogonal rotation factor to the point makes it rotate to the OR_4 axis. Thus, $e^{i_3\theta_3}i_4 = i_4$. On the other hand, applying the rotation factor $e^{i_3\theta_3}$ to a point represented by i_4 in the OR_4 axis will not move the point. Thus, $i_4e^{i_3\theta_3} = i_4$.

Further, from the 4D perspective, the 2nd-dimension axis-plane, which is formed by the RE and OR_2 axes in Figure 6 for the 3D, now appears 1D in Figure 7 for the 4D. Logically, in Figure 7, the point represented by the i_2-based rotation factor $e^{i_2\theta_2}$ is in the OR_2 axis. Applying the i_4 orthogonal rotation factor to the point makes it rotate to the OR_4 axis. Thus, $e^{i_2\theta_2}i_4 = i_4$. On the other hand, applying the rotation factor $e^{i_2\theta_2}$ to a point represented by i_4 in the OR_4 axis will not move the point. Thus, $i_4e^{i_2\theta_2} = i_4$.

From the above, it follows that

$$e^{i_3\theta_3}i_4 = i_4e^{i_3\theta_3} = e^{i_2\theta_2}i_4 = i_4e^{i_2\theta_2} = i_4 \tag{55}$$

With angle $\frac{\pi}{2}$ for orthogonal rotation factors, Equation (55) leads to

$$i_3i_4 = i_4i_3 = i_2i_4 = i_4i_2 = i_4 \tag{56}$$

Equations (55) and (56) indicate that the orthogonal rotation factor i_4 is multiplication commutative with the orthogonal rotation factors and rotation factors of i_2, $e^{i_2\theta_2}$, i_3, and $e^{i_3\theta_3}$ in the lower dimensions, and the multiplication result is always equal to the i_4 itself.

6.3. n-dimensional rotation factors

Similarly, in Figure 8 for the n-dimensional space, the $\mathrm{OR_n}$ axis is orthogonal to the (n-1)th dimension axis-plane, which is formed by the $\mathrm{OR_{n-2}}$ and $\mathrm{OR_{n-1}}$ axes. The point represented by the i_{n-1} based rotation factor $e^{i_{n-1}\theta_{n-1}}$ is in the (n-1)th dimension axis-plane. Applying the i_n orthogonal rotation factor to the point makes it rotate to the $\mathrm{OR_n}$ axis. Thus, $e^{i_{n-1}\theta_{n-1}} i_n = i_n$. On the other hand, applying the rotation factor $e^{i_{n-1}\theta_{n-1}}$ to a point represented by i_n in the $\mathrm{OR_n}$ axis will not move the point. Thus, $i_n e^{i_{n-1}\theta_{n-1}} = i_n$.

Further, from the n-dimensional perspective, in Figure 8, the point represented by the i_{n-2} based rotation factor $e^{i_{n-2}\theta_{n-2}}$ is in the $\mathrm{OR_{n-2}}$ axis. Applying the i_n orthogonal rotation factor to the point makes it rotate to the $\mathrm{OR_n}$ axis. Thus, $e^{i_{n-2}\theta_{n-2}} i_n = i_n$. On the other hand, applying the rotation factor $e^{i_{n-2}\theta_{n-2}}$ to a point represented by i_n in the $\mathrm{OR_n}$ axis will not move the point. Thus, $i_n e^{i_{n-2}\theta_{n-2}} = i_n$.

Still further, by the same token, in the n-dimensional coordinate system, all axes are mutually orthogonal to each other. For a jth dimension with j lower than n-2, applying the i_n orthogonal rotation factor to a point represented by $e^{i_j\theta_j}$ in the jth dimension makes the point rotate to the $\mathrm{OR_n}$ axis. Thus, $e^{i_j\theta_j} i_n = i_n$. On the other hand, applying the rotation factor $e^{i_j\theta_j}$ to a point represented by i_n in the $\mathrm{OR_n}$ axis will not move the point. Thus, $i_n e^{i_j\theta_j} = i_n$.

And still further, applying the i_n orthogonal rotation factor twice to the point $\mathrm{p_n}$ makes the point rotate by angle π in the plane formed by the point $\mathrm{p_n}$ and the $\mathrm{OR_n}$ axis, with the resultant position vector's direction being reversed. Thus, $\mathrm{p_n} i_n^2 = -\mathrm{p_n}$. That is, $i_n^2 = -1$.

From the above and with the generalization, it follows that

$$e^{i_j\theta_j} i_n = i_n e^{i_j\theta_j} = i_n \tag{57}$$

where $2 \leq j < n$.

$$i_j i_n = i_n i_j = i_n \tag{58}$$

where $2 \leq j < n$.

$$i_j^2 = -1 \tag{59}$$

where $2 \leq j \leq n$.

Equations (53)-(54), (55)-(56), and (57)-(58) indicate that the orthogonal rotation factors i_j and the rotation factors $e^{i_j\theta_j}$ in the n-dimensional spherical coordinate system represented by Equation (39) are multiplication commutative.

It is noted that the above results can also be obtained by the spherical-to-Cartesian transformation in (43). Take (57) for example. The position number $e^{i_j\theta_j} i_n = e^{i_j\theta_j} e^{i_n \frac{\pi}{2}}$ means that in (43), $r_n = 1$, $\theta_j = \theta_j$, $\theta_n = \frac{\pi}{2}$ and all other angles (θ_k with $2 \leq k \leq n-1$ excluding $k = j$) are 0. Thus, all E_j in (43) become 0 except for E_n, which is 1. That is, $\mathrm{p_n}$ $= i_n$, which is consistent with (57).

6.4. Obtaining n-dimensional spherical-to-Cartesian transformation by rotation factor multiplication algebra

Next, the spherical-to-Cartesian transformation equation in (43) is obtained by rotation factor multiplication algebra.

In Equation (39), denote

$$Q_j = \prod_{k=2}^{j} e^{i_k \theta_k} \tag{60}$$

Equation (39) becomes

$$\frac{\mathrm{p_n}}{\mathrm{r_n}} = Q_n \tag{61}$$

The result in (57) means

$$Q_{j-1} i_j = i_j \tag{62}$$

With (62), it follows that

$$Q_n = Q_{n-1} e^{i_n \theta_n} = Q_{n-1}(cos(\theta_n) + i_n sin(\theta_n)) = Q_{n-1} cos(\theta_n) + i_n sin(\theta_n) \tag{63}$$

With (63), it follows that

$$Q_{n-1} = Q_{n-2} cos(\theta_{n-1}) + i_{n-1} sin(\theta_{n-1}) \tag{64}$$

and

$$Q_{n-2} = Q_{n-3} cos(\theta_{n-2}) + i_{n-2} sin(\theta_{n-2}) \tag{65}$$

Q_n may be obtained by continuing the iteration process and combining the iteration results. Inserting the obtained Q_n into (61) leads to Equation (43).

7. Summary

Based on the existence of the set of rotation factors and the concept of rotation factors for rotating position vectors and numbers with positioning directions, the constructions of three and higher n-dimensional complex number spherical coordinate systems are realized.

7.1. Methodology of increment in dimensional levels for coordinate system construction

Starting from one-dimension represented by a real number axis, a 2nd-dimension orthogonal axis is added to form a plane where the associated rotation factors rotate real numbers in the real axis into the plane. Then, the 3rd-dimension orthogonal axis is added to form a 3rd-dimension where the associated rotation factors rotate points in the 2nd-dimension into the current higher dimension. And then, the 4th-dimension orthogonal axis is added to form a 4th-dimension where the associated rotation factors rotate points in the 3rd-dimension into the current higher dimension. The process can be continued and generalized to the nth-dimension.

7.2. Methodology for projections of position vectors in four-dimensional and higher n-dimensional spaces

By examining the patterns of projections of position vectors in the two-dimensional plane and three-dimensional space, the projection patterns are then extended to higher dimensional spaces. A projection chain is established where a position vector that makes a projection to a position vector in a lower dimension becomes the projection of a position vector in a higher dimension. The projection patterns in the chain are consistent throughout with the same mathematical formulas from nth dimension, down to 4th, 3rd, and 2nd.

The projections are helpful in understanding the geometric representation of position vectors in four-dimensional and higher n-dimensional spaces, and provide not only the qualitative insights but also the quantitative precise results.

7.3. n-dimensional complex number (position number) spherical coordinate system and transformations between Cartesian and spherical coordinates

Once the coordinate axis construction and the projections of position vectors are established, the existence of the rotation factors and their associate rotation properties are the natural fit for expressing the n-dimensional complex number spherical coordinate system with inherent simplicity and succinctness.

The generalizations of the transformations from spherical coordinates to Cartesian coordinates and vice versa are achieved by obtaining the relevant transformation formulas for each dimensional level through the projection chain.

7.4. Multiplication algebra and interactions between rotation factors

One method for obtaining the multiplication and interactions between orthogonal rotation factors and rotation factors in the same and across dimensional levels is by the geometric representation where rotation factors are applied to position vectors and the resultant rotations are examined.

Another method is by the transformation from spherical coordinates to Cartesian coordinates as the transformation equation contains the interaction information.

The multiplication algebra method may also be used to obtain the transformation to the Cartesian coordinate system from the spherical system. The three methods produce results that are consistent with each other.

7.5. Motion generation characteristics of rotation factors

The construction of the n-dimensional complex number spherical and Cartesian systems is made possible by the rotation factors.

The existence of the set of the rotation factors along with the associated rotation properties is a gift from Nature. The position rotation represents motion. A rotation factor generates a rotation or motion when it is applied to a position point or to a vector. It represents an active force and empowerment tool for encouragement of the novel utilization.

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